Partial Differential Equations: Midterm Exam

Aletta Jacobshal 01, Tuesday 7 March 2017, 14:00–16:00
Exam duration: 2 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of your exam sheet and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explain why the conditions for using such results are satisfied.
- 10 points are "free". There are 4 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (20 points)

Consider the eigenvalue problem

$$-X''(x) = \lambda X(x), \tag{1}$$

with $0 \leq x \leq 1$, where $X(x)$ satisfies the boundary conditions

$$X'(0) + X(0) = X'(1) - X(1) = 0.$$

We consider only the case of negative eigenvalues, $\lambda = -\gamma^2$, with $\gamma > 0$. Prove that γ must satisfy the equation

$$\tanh \gamma = \frac{2\gamma}{\gamma^2 + 1}. \tag{2}$$

Using the graphical method, show that there is exactly one negative eigenvalue.

NB: In this question you may NOT consider as known either one of the formulas for negative or positive eigenvalues from the corresponding theory; you are being asked to prove Eq. (2) "from scratch", starting with the general solution of the ODE in Eq. (1).

Solution

We start by considering the general solution of $X'' = \gamma^2 X$, given by

$$X = C \cosh \gamma x + D \sinh \gamma x.$$

The derivative is

$$X' = \gamma C \sinh \gamma x + \gamma D \cosh \gamma x.$$

Then the boundary conditions become

$$X'(0) + X(0) = C + \gamma D = 0 \Rightarrow C = -\gamma D,$$

and

$$X'(1) - X(1) = \gamma C \sinh \gamma + \gamma D \cosh \gamma - C \cosh \gamma - D \sinh \gamma = 0,$$

giving

$$D(-\gamma^2 \sinh\gamma + 2\gamma\cosh\gamma - \sinh\gamma) = 0.$$

We exclude the case $D = 0$ because then also $C = 0$. Then we get

$$(\gamma^2 + 1)\sinh\gamma = 2\gamma\cosh\gamma,$$

and, finally,

$$\tanh\gamma = \frac{2\gamma}{\gamma^2 + 1} =: g(\gamma).$$

We plot the graphs of the two functions, $\tanh\gamma$ and $g(\gamma)$.

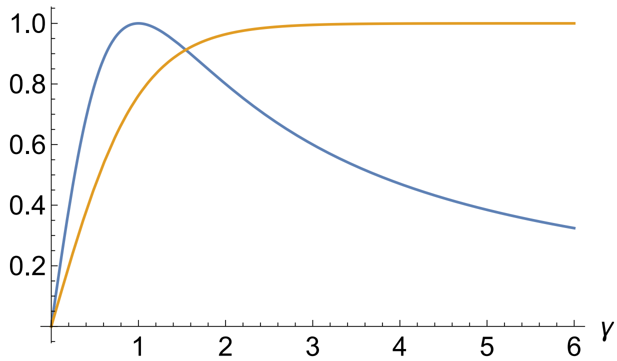
We have

$$g'(\gamma) = \frac{2(1-\gamma^2)}{(\gamma^2+1)^2}.$$

From here we find that $g(\gamma)$ attains the maximum value $g(1) = 1$ at $\gamma = 1$ and that at $\gamma = 0$ we have $g'(0) = 2$. Moreover, $\lim_{\gamma\to\infty} g(\gamma) = 0$.

The function $\tanh\gamma$ is increasing with $\tanh\gamma < 1$ and $\lim_{\gamma\to\infty}\tanh\gamma = 1$. Moreover, at $\gamma = 0$ we have $\tanh'(0) = 1 < g'(0)$.

Therefore, we have the situation shown in the graph below, implying that there is exactly one intersection between the two graphs.



Question 2 (30 points)

(a) (15 points) Consider the non-homogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad -\infty < x < \infty,\ t > 0,$$

satisfying the homogeneous initial conditions $u(x,0) = 0$ and $u_t(x,0) = 0$. Show that the function

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right) ds,$$

is a solution to the given problem.

Hint: Computations become easier by introducing a function $F(x,t)$ such that $F_x(x,t) = f(x,t)$.

Solution

We first note that the given function satisfies

$$u(x,0) = \frac{1}{2c}\int_0^0 \left(\int_{x+cs}^{x-cs} f(y,s)\,dy\right) ds = 0.$$

Then we compute u_t. This becomes easier if we write $F(x,t)$ for the x-antiderivative of $f(x,t)$, that is, $F_x = f$. Then

$$u(x,t) = \frac{1}{2c}\int_0^t \left(F(x+c(t-s),s) - F(x-c(t-s),s)\right) ds.$$

We have

$$\begin{aligned} u_t(x,t) &= \frac{1}{2c}\left(F(x+c(t-t),t) - F(x-c(t-t),t)\right) \\ &\quad + \frac{1}{2c}\int_0^t \left(cF_x(x+c(t-s),s) + cF_x(x-c(t-s),s)\right) ds \\ &= \frac{1}{2c}\int_0^t \left(cf(x+c(t-s),s) + cf(x-c(t-s),s)\right) ds. \end{aligned}$$

Therefore, for $t = 0$ we find that

$$u_t(x,0) = \frac{1}{2c}\int_0^0 \left(cf(x-cs),s) + cf(x+cs,s)\right) ds = 0.$$

Then we compute

$$\begin{aligned} u_{tt}(x,t) &= \frac{1}{2c}\left(cf(x+c(t-t),t) + cf(x-c(t-t),t)\right) \\ &\quad + \frac{1}{2c}\int_0^t \left(c^2 f_x(x+c(t-s),s) - c^2 f_x(x-c(t-s),s)\right) ds \\ &= f(x,t) + \frac{c}{2}\int_0^t \left(f_x(x+c(t-s),s) - f_x(x-c(t-s),s)\right) ds. \end{aligned}$$

Moreover,

$$\begin{aligned} u_x(x,t) &= \frac{1}{2c}\int_0^t \left(F_x(x+c(t-s),s) - F_x(x-c(t-s),s)\right) ds \\ &= \frac{1}{2c}\int_0^t \left(f(x+c(t-s),s) - f(x-c(t-s),s)\right) ds, \end{aligned}$$

and

$$u_{xx}(x,t) = \frac{1}{2c}\int_0^t \left(f_x(x+c(t-s),s) - f_x(x-c(t-s),s)\right) ds.$$

This shows that

$$u_{tt} = f(x,t) + c^2 u_{xx}.$$

(b) (15 points) Solve the non-homogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad -\infty < x < \infty,\ t > 0,$$

satisfying now the **non-homogeneous** initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = 0$.

Hint: The equation (including the boundary conditions) is linear.

Solution

The solution to the homogeneous equation

$$u_{tt} = c^2 u_{xx},$$

with $u(x,0) = \phi(x)$ and $u_t(x,0) = 0$ is given by

$$u_1(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)].$$

The solution to the non-homogeneous equation

$$u_{tt} = c^2 u_{xx} + f(x,t),$$

with $u(x,0) = 0$ and $u_t(x,0) = 0$ is given by

$$u_2(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right) ds,$$

as it was shown in the previous subquestion.

Then we define $u = u_1 + u_2$ and we have the following. First,

$$u_{tt} = (u_1)_{tt} + (u_2)_{tt} = c^2(u_1)_{xx} + c^2(u_2)_{xx} + f = c^2 u_{xx} + f.$$

Then

$$u(x,0) = u_1(x,0) + u_2(x,0) = \phi(x) + 0 = \phi(x),$$

and

$$u_t(x,0) = (u_1)_t(x,0) + (u_2)_t(x,0) = 0 + 0 = 0.$$

Therefore, the function

$$u(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right) ds$$

solves the given problem.

Question 3 (20 points)

Solve the equation $u_x + u_y = u$ where the solution satisfies $u(x,0) = x^2$.

Solution

The coordinate method from the book gives

$$s = x + y, \quad t = x - y.$$

We then have

$$u_x + u_y = \left(u_s \frac{\partial s}{\partial x} + u_t \frac{\partial t}{\partial x}\right) + \left(u_s \frac{\partial s}{\partial y} + u_t \frac{\partial t}{\partial y}\right) = 2u_s.$$

Therefore, the equation becomes

$$u_s = \frac{u}{2},$$

which has solution

$$u = g(t)e^{\frac{1}{2}s}.$$

Therefore, the solution in terms of the original coordinates is

$$u = g(x-y)e^{\frac{1}{2}(x+y)}.$$

Then for $y = 0$ we have

$$g(x)e^{x/2} = x^2,$$

implying

$$g(x) = x^2e^{-x/2}.$$

Finally,

$$u = (x-y)^2e^{-\frac{1}{2}(x-y)}e^{\frac{1}{2}(x+y)} = (x-y)^2e^y.$$

Question 4 (20 points)

Use the method of separation of variables to solve the equation

$$u_t = ku_{xx} - u,$$

where $k > 0$, $0 \leq x \leq \ell$, $t \geq 0$, and the solution satisfies the boundary conditions $u(0,t) = u(\ell,t) = 0$ and the initial condition

$$u(x,0) = 2\sin\left(\frac{\pi x}{\ell}\right) + \frac{1}{5}\sin\left(\frac{3\pi x}{\ell}\right).$$

NB: In this question you can use without proof the formulas for the eigenvalues and eigenfunctions of any eigenvalue problem discussed in the lectures.

Solution

We take the separated solution

$$u(x,t) = X(x)T(t).$$

Then we have

$$XT' = kX''T - XT,$$

and

$$\frac{T'}{kT} = \frac{X''}{X} - \frac{1}{k}.$$

To be able to easily use the known results we rewrite

$$\frac{T'}{kT} + \frac{1}{k} = \frac{X''}{X} = -\lambda.$$

From the x-part we get the eigenvalue problem

$$-X'' = \lambda X, \quad X(0) = X(\ell) = 0.$$

The eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 1, 2, 3, \ldots,$$

while the eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right), \quad n = 1, 2, 3, \ldots.$$

The t-part gives the ODE

$$\frac{T'}{T} = -k\lambda - 1,$$

with solution

$$T = Ae^{-(k\lambda+1)t}.$$

Therefore, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(k\lambda_n+1)t} \sin\left(\frac{n\pi x}{\ell}\right).$$

For $t = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right),$$

which implies $A_1 = 2$, $A_3 = 1/5$, and all other A_n are zero. Therefore,

$$u(x,t) = 2e^{-(k\lambda_1+1)t} \sin\frac{\pi x}{\ell} + \frac{1}{5} e^{-(k\lambda_3+1)t} \sin\frac{3\pi x}{\ell},$$

or

$$u(x,t) = 2e^{-(k(\pi/\ell)^2+1)t} \sin\frac{\pi x}{\ell} + \frac{1}{5} e^{-(9k(\pi/\ell)^2+1)t} \sin\frac{3\pi x}{\ell}.$$